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On the Fragmentation Function for Heavy Quarks in e^+e^- collisions

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Abstract

We use a recent $\mathcal{O}(\alpha_s^2)$ calculation of the differential cross section for the production of heavy quarks in e^+e^- annihilation to compute a few moments of the heavy quark single-inclusive production cross section. We verify that, contrary to some recent claims, the leading and next-to-leading logarithmic terms in this cross section are correctly given by the standard NLO fragmentation function formalism for heavy quark production.

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1 Introduction

It is known that inclusive heavy quark production is a calculable process in perturbative QCD, since the heavy quark mass acts as a cut-off for the final state collinear singularities. Thus, the process

$$e^+e^- \to Z/\gamma \to Q + X \; , \tag{1.1}$$

where Q is the heavy quark and X is anything else, is calculable. Its cross section can be expressed as a power expansion in the strong coupling constant

$$\frac{d\sigma}{dx}(x,E,m) = \sum_{n=0}^{\infty} a^{(n)}(x,E,m,\mu)\,\bar{\alpha}_s^n(\mu) \; , \tag{1.2}$$

where E is the centre-of-mass energy, m is the mass of the heavy quark, μ is the renormalization scale, and

$$\bar{\alpha}_s(\mu) = \frac{\alpha_s(\mu)}{2\pi} \; . \tag{1.3}$$

As usual we define

$$x = \frac{2\,p\cdot q}{q^2} \; , \tag{1.4}$$

where q and p are the four-momenta of the intermediate virtual boson and of the final heavy quark Q. The cross section (1.2), normalized to the total cross section, is sometimes referred to as the heavy quark fragmentation function in e^+e^- annihilation. When E/m is not too large, the truncation of eq. (1.2) at some fixed order in the coupling can be used to compute the cross section. On the other hand, if $E \gg m$, the $n^{\rm th}$ order coefficient of the expansion will in general contain up to n powers of $\log(E/m)$, thereby spoiling the convergence of the expansion. These large logarithms can be resummed, according to the method described in ref. [1]. First of all, since $\log(E/m)$ is large, one is entitled to neglect the terms that are suppressed by powers of m/E. One then observes that, in this limit, the inclusive heavy quark cross section must satisfy the factorization theorem formula

$$\frac{d\sigma}{dx}(x,E,m) = \sum_i \int_x^1 \frac{dz}{z}\,\frac{d\hat{\sigma}_i}{dz}(z,E,\mu)\;\hat{D}_i\left(\frac{x}{z},\mu,m\right) \; , \tag{1.5}$$

where $d\hat{\sigma}_i(z,E,\mu)/dz$ are the $\overline{\rm MS}$-subtracted partonic cross sections for producing the parton i, and $\hat{D}_i(x,\mu,m)$ are the $\overline{\rm MS}$ fragmentation functions for the parton i into the heavy quark Q. In order for eq. (1.5) to hold, it is essential that one uses a renormalization scheme where the heavy flavour is treated as a light one, like the pure $\overline{\rm MS}$ scheme. Thus $d\hat{\sigma}_i(z,E,\mu)/dz$ has a perturbative expansion in terms of α_s with n_f flavours, where n_f includes the heavy one. The scale μ is the factorization and renormalization scale. It should be chosen of the order of E, in order to avoid the appearance of large logarithms of E/μ in the partonic cross section. The $\overline{\rm MS}$ fragmentation functions $\hat{D}_i$ obey the Altarelli-Parisi evolution equations:

$$\frac{d\hat{D}_i}{d\log\mu^2}(x,\mu,m) = \sum_j \int_x^1 \frac{dz}{z}\, P_{ij}\left(\frac{x}{z},\bar{\alpha}_s(\mu)\right)\, \hat{D}_j(z,\mu,m)\;. \tag{1.6}$$

The Altarelli-Parisi splitting functions P_{ij} have the perturbative expansion

$$P_{ij}\Big(x,\bar{\alpha}_s(\mu)\Big) = \bar{\alpha}_s(\mu)P_{ij}^{(0)}(x) + \bar{\alpha}_s^2(\mu)P_{ij}^{(1)}(x) + \mathcal{O}(\bar{\alpha}_s^3)\;, \tag{1.7}$$

where $P_{ij}^{(0)}$ are given in ref. [2] and $P_{ij}^{(1)}$ have been computed in refs. [4]-[7]. The only missing ingredients for the calculation of the inclusive cross section are the initial conditions for the $\overline{\rm MS}$ fragmentation functions. These were obtained at the NLO level in ref. [1] by matching the $\mathcal{O}(\bar{\alpha}_s)$ direct calculation of the process (i.e. formula (1.2)) with the expansion of formula (1.5) at order $\bar{\alpha}_s$. They have the form

$$\begin{aligned}
\hat{D}_Q(x,\mu_0,m) &= \delta(1-x) + \bar{\alpha}_s(\mu_0)\, d_Q^{(1)}(x,\mu_0,m) + \mathcal{O}(\bar{\alpha}_s^2) \\
\hat{D}_g(x,\mu_0,m) &= \bar{\alpha}_s(\mu_0)\, d_g^{(1)}(x,\mu_0,m) + \mathcal{O}(\bar{\alpha}_s^2)\;,
\end{aligned} \tag{1.8}$$

all the other components being of order $\bar{\alpha}_s^2$. Thus, in order to compute the NLO resummed expansion, one takes the initial conditions eqs. (1.8), at a value of μ_0 of order m, evolves them at the scale μ (taken to be of order E), and then applies formula (1.5), using a NLO expression for the partonic cross section

$$\frac{d\hat{\sigma}_i}{dx}(x,E,\mu) = \hat{a}_i^{(0)}(x) + \hat{a}_i^{(1)}(x,E,\mu)\,\bar{\alpha}_s(\mu) + \mathcal{O}(\bar{\alpha}_s^2)\;. \tag{1.9}$$

For example, if the parton i is the heavy quark itself, one gets

$$\frac{d\hat{\sigma}_Q}{dx}(x,E,\mu) = \delta(1-x) + \hat{a}_Q^{(1)}(x,E,\mu)\,\bar{\alpha}_s(\mu) + \mathcal{O}(\bar{\alpha}_s^2)\;, \tag{1.10}$$

where we have normalized the cross section to 1 at zeroth order in the strong coupling constant.

The procedure outlined above guarantees that all terms of the form $(\bar{\alpha}_s L)^n$ (leading order) and $\bar{\alpha}_s(\bar{\alpha}_s L)^n$ (next-to-leading order), where L is the large logarithm, are included correctly in the resummed formula. Observe that, at the NLO level, the scale that appears in $\bar{\alpha}_s$ in eqs. (1.8) and (1.9) could be changed by factors of order 1, since this amounts to a correction of order $\bar{\alpha}_s^2$. However, one cannot set $\mu_0 = \mu$ in eqs. (1.8) (or $\mu = m$ in formula (1.9)), since this amounts to a correction of order $\bar{\alpha}_s^2 L$, and thus it would spoil the validity of the resummation formula at the NLO level.

There is essentially no room for these considerations to fail. They are a consequence of the factorization theorem for fragmentation functions, which is quite well established [3, 4].

The validity of this procedure has however been questioned by Kniehl et al. in ref. [8]. In their procedure, the heavy quark short distance cross section is replaced by

$$\frac{d\hat{\sigma}'_Q}{dx}(x,E,\mu) = \delta(1-x) + \bar{\alpha}_s(E)\,\left[\hat{a}_Q^{(1)}(x,E,\mu) + d_Q^{(1)}(x,\mu_0,m)\right] + \mathcal{O}(\bar{\alpha}_s^2)\;, \tag{1.11}$$

and the initial condition by

$$\hat{D}'_Q(x) = \delta(1-x)\;, \tag{1.12}$$

which is to be evolved from the scale μ_0 to the scale μ using the NLO $\overline{\rm MS}$ evolution equations. This procedure differs at the NLO level from the standard procedure advocated in ref. [1]. The difference starts to show up in the terms of order $\bar{\alpha}_s^2 L$. Recently, we have completed a calculation of heavy quark inclusive production at order $\bar{\alpha}_s^2$ [10]. Using this calculation, we are in a position to verify explicitly the approach of ref. [1], and thereby dismiss the approach of ref. [8]. In the next section we describe the procedure we followed in detail.

2 Calculation

Instead of dealing with the realistic case of Z/γ decay, we perform the calculation for a hypothetical vector boson V that couples only to the heavy quark with vectorial coupling.

We introduce the following notation for the Mellin transform:

$$f(N) \equiv \int_0^1 dx\, x^{N-1} f(x)\;. \tag{2.1}$$

We adopt the convention that, when N appears instead of x as the argument of a function, we are actually referring to the Mellin transform of the function. This notation is somewhat improper, but it should not generate confusion in the following, since we will be working only with Mellin transforms. The Mellin transform of the factorization formula (1.5) is given by

$$\sigma(N,E,m) = \sum_i \hat{\sigma}_i(N,E,\mu)\ \hat{D}_i(N,\mu,m)\ , \tag{2.2}$$

where

$$\sigma(N,E,m) = \int_0^1 dx\, x^{N-1} \frac{d\sigma}{dx}(x,E,m)\ , \tag{2.3}$$

and a similar one for $\hat{\sigma}_i(N,E,\mu)$, and the Mellin transform of the Altarelli-Parisi evolution equation (1.6) is

$$\frac{d\hat{D}_i(N,\mu,m)}{d\log\mu^2} = \sum_j \bar{\alpha}_s(\mu)\left[P_{ij}^{(0)}(N) + P_{ij}^{(1)}(N)\,\bar{\alpha}_s(\mu) + \mathcal{O}(\bar{\alpha}_s^2)\right]\hat{D}_j(N,\mu,m)\ . \tag{2.4}$$

We want to obtain an expression for $\sigma(N,E,m)$ valid at the second order in $\bar{\alpha}_s$. Thus, we need the solution of eq. (2.4), with initial condition at $\mu=\mu_0$, accurate at order $\bar{\alpha}_s^2$. This is easily done by rewriting eq. (2.4) as an integral equation

$$\begin{aligned}\hat{D}_i(N,\mu,m) &= \hat{D}_i(N,\mu_0,m)\\ &+\sum_j \int_{\mu_0}^{\mu} d\log\mu'^2\ \bar{\alpha}_s(\mu')\left[P_{ij}^{(0)}(N) + P_{ij}^{(1)}(N)\,\bar{\alpha}_s(\mu')\right]\hat{D}_j(N,\mu',m)\ .\end{aligned} \tag{2.5}$$

The terms proportional to $\bar{\alpha}_s^2$ can be evaluated at any scale (μ or μ_0), the difference being of order $\bar{\alpha}_s^3$. Factors involving a single power of $\bar{\alpha}_s$ can instead be expressed in terms of $\bar{\alpha}_s(\mu_0)$ using the renormalization group equation

$$\begin{aligned}\bar{\alpha}_s(\mu') &= \bar{\alpha}_s(\mu_0) - 2\pi\, b_0\, \bar{\alpha}_s^2(\mu_0)\ \log\frac{\mu'^2}{\mu_0^2} + \mathcal{O}(\bar{\alpha}_s^3(\mu_0))\\ b_0 &= \frac{11C_A - 4\,n_f\,T_F}{12\pi}\ ,\end{aligned} \tag{2.6}$$

where n_f is the number of flavours including the heavy one. Equation (2.5) then becomes

$$\begin{aligned}\hat{D}_i(N,\mu,m) &= \hat{D}_i(N,\mu_0,m) + \sum_j \int_{\mu_0}^{\mu} d\log\mu'^2\ \bar{\alpha}_s(\mu_0) P_{ij}^{(0)}(N)\hat{D}_j(N,\mu',m)\\ &+ \sum_j \bar{\alpha}_s^2(\mu_0) P_{ij}^{(1)}(N)\hat{D}_j(N,\mu_0,m)\ \log\frac{\mu^2}{\mu_0^2}\\ &- 2\pi\, b_0 \sum_j \bar{\alpha}_s^2(\mu_0) P_{ij}^{(0)}(N)\hat{D}_j(N,\mu_0,m)\ \frac{1}{2}\log^2\frac{\mu^2}{\mu_0^2}\ .\end{aligned} \tag{2.7}$$

We now need to express $\hat{D}_j(N,\mu',m)$ on the right-hand side of the above equation as a function of the initial condition, with an accuracy of order $\bar{\alpha}_s$. This is simply done by iterating the above equation once, keeping only the first two terms on the right-hand side. Our final result is then

$$\begin{aligned}
\hat{D}_i(N,\mu,m) &= \hat{D}_i(N,\mu_0,m) + \sum_j \bar{\alpha}_s(\mu_0) P_{ij}^{(0)}(N) \hat{D}_j(N,\mu_0,m)\ \log\frac{\mu^2}{\mu_0^2} \\
&+ \sum_{kj} \bar{\alpha}_s^2(\mu_0) P_{ik}^{(0)}(N)\ P_{kj}^{(0)}(N) \hat{D}_j(N,\mu_0,m)\ \frac{1}{2}\log^2\frac{\mu^2}{\mu_0^2} \\
&+ \sum_j \bar{\alpha}_s^2(\mu_0) P_{ij}^{(1)}(N) \hat{D}_j(N,\mu_0,m)\ \log\frac{\mu^2}{\mu_0^2} \\
&- 2\pi\, b_0 \sum_j \bar{\alpha}_s^2(\mu_0) P_{ij}^{(0)}(N) \hat{D}_j(N,\mu_0,m)\ \frac{1}{2}\log^2\frac{\mu^2}{\mu_0^2}\,. \qquad (2.8)
\end{aligned}$$

Since the initial condition is

$$\hat{D}_j(N,\mu_0,m) = \delta_{jQ} + \bar{\alpha}_s(\mu_0)\, d_j^{(1)}(N,\mu_0,m) + \mathcal{O}(\bar{\alpha}_s^2(\mu_0))\,, \qquad (2.9)$$

eq. (2.8) becomes, with the required accuracy:

$$\begin{aligned}
\hat{D}_i(N,\mu,m) &= \delta_{iQ} + \bar{\alpha}_s(\mu_0)\, d_i^{(1)}(N,\mu_0,m) + \bar{\alpha}_s(\mu_0) P_{iQ}^{(0)}(N)\ \log\frac{\mu^2}{\mu_0^2} \\
&+ \sum_j \bar{\alpha}_s^2(\mu_0) P_{ij}^{(0)}(N) d_j^{(1)}(N,\mu_0,m)\ \log\frac{\mu^2}{\mu_0^2} \\
&+ \sum_k \bar{\alpha}_s^2(\mu_0) P_{ik}^{(0)}(N)\ P_{kQ}^{(0)}(N)\ \frac{1}{2}\log^2\frac{\mu^2}{\mu_0^2} \\
&+ \bar{\alpha}_s^2(\mu_0) P_{iQ}^{(1)}(N)\ \log\frac{\mu^2}{\mu_0^2} \\
&- 2\pi\, b_0\, \bar{\alpha}_s^2(\mu_0) P_{iQ}^{(0)}(N)\ \frac{1}{2}\log^2\frac{\mu^2}{\mu_0^2}\,. \qquad (2.10)
\end{aligned}$$

Re-expressing $\bar{\alpha}_s(\mu_0)$ in terms of μ, we get

$$\begin{aligned}
\hat{D}_i(N,\mu,m) &= \delta_{iQ} + \bar{\alpha}_s(\mu)\, d_i^{(1)}(N,\mu_0,m) + 2\pi\, b_0\, \bar{\alpha}_s^2(\mu)\, d_i^{(1)}(N,\mu_0,m)\ \log\frac{\mu^2}{\mu_0^2} \\
&+ \bar{\alpha}_s(\mu) P_{iQ}^{(0)}(N)\ \log\frac{\mu^2}{\mu_0^2} + \sum_j \bar{\alpha}_s^2(\mu) P_{ij}^{(0)}(N) d_j^{(1)}(N,\mu_0,m)\ \log\frac{\mu^2}{\mu_0^2}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \sum_k \bar{\alpha}_s^2(\mu) P_{ik}^{(0)}(N)\; P_{kQ}^{(0)}(N)\; \frac{1}{2}\log^2\frac{\mu^2}{\mu_0^2} + \bar{\alpha}_s^2(\mu) P_{iQ}^{(1)}(N)\; \log\frac{\mu^2}{\mu_0^2} \\
&+ \quad \pi\, b_0\, \bar{\alpha}_s^2(\mu) P_{iQ}^{(0)}(N)\; \log^2\frac{\mu^2}{\mu_0^2}\,. \qquad (2.11)
\end{aligned}
$$

The partonic cross sections are given by

$$
\hat{\sigma}_i(N,E,\mu) = \delta_{iQ} + \delta_{i\overline{Q}} + \bar{\alpha}_s(\mu)\,\hat{a}_i^{(1)}(N,E,\mu) + \mathcal{O}(\bar{\alpha}_s^2(\mu))\;, \qquad (2.12)
$$

where $\hat{a}_i^{(1)}$ vanishes unless i is either Q, $\overline{Q}$ or g. Thus, combining eq. (2.12) with eq. (2.11) according to eq. (2.2), we obtain

$$
\begin{aligned}
\sigma(N,E,m) \;&=\; 1 + \bar{\alpha}_s(\mu)\left[\hat{a}_Q^{(1)}(N,E,\mu) + d_Q^{(1)}(N,\mu_0,m) + P_{QQ}^{(0)}(N)\log\frac{\mu^2}{\mu_0^2}\right] \\
&+\; \bar{\alpha}_s^2(\mu)\Bigg\{\sum_i \hat{a}_i^{(1)}(N,E,\mu) P_{iQ}^{(0)}\;\log\frac{\mu^2}{\mu_0^2} + 2\pi\, b_0\, d_Q^{(1)}(N,\mu_0,m)\;\log\frac{\mu^2}{\mu_0^2} \\
&+\; \sum_j \left[P_{Qj}^{(0)}(N) + P_{\overline{Q}j}^{(0)}(N)\right]\; d_j^{(1)}(N,\mu_0,m)\;\log\frac{\mu^2}{\mu_0^2} \\
&+\; \sum_k \left[P_{Qk}^{(0)}(N) + P_{\overline{Q}k}^{(0)}(N)\right]\; P_{kQ}^{(0)}(N)\;\frac{1}{2}\log^2\frac{\mu^2}{\mu_0^2} \\
&+\; \left[P_{QQ}^{(1)}(N) + P_{\overline{Q}Q}^{(1)}(N)\right]\log\frac{\mu^2}{\mu_0^2} + \pi\, b_0\, P_{QQ}^{(0)}(N)\;\log^2\frac{\mu^2}{\mu_0^2}\Bigg\}\,. \qquad (2.13)
\end{aligned}
$$

The above formula should accurately describe the terms of order $\bar{\alpha}_s L$, $\bar{\alpha}_s$, $\bar{\alpha}_s^2 L^2$ and $\bar{\alpha}_s^2 L$. Terms of order $\bar{\alpha}_s^2$, without logarithmic enhancement, are not accurately given by the fragmentation formalism at NLO level, and have consistently been neglected.

The lowest order splitting functions are given by

$$
\begin{aligned}
P_{QQ}^{(0)}(N) \;&=\; C_F\left[\frac{3}{2} + \frac{1}{N(N+1)} - 2\,S_1(N)\right]\,, \\
P_{Qg}^{(0)}(N) \;&=\; P_{\overline{Q}g}^{(0)}(N) = C_F\left[\frac{2+N+N^2}{N(N^2-1)}\right]\,, \\
P_{gQ}^{(0)}(N) \;&=\; T_F\left[\frac{2+N+N^2}{N(N+1)(N+2)}\right]\,, \qquad (2.14)
\end{aligned}
$$

where, restricting ourselves to integer values of N,

$$
S_1(N) = \sum_{j=1}^{N}\frac{1}{j}\,. \qquad (2.15)
$$

$P^{(1)}_{QQ}(N)$ and $P^{(1)}_{\overline{Q}Q}(N)$ are given by

$$P^{(1)}_{QQ}(N) = P^{\rm NS}_{QQ}(N) + P_{q'q}(N)\,, \quad P^{(1)}_{\overline{Q}Q}(N) = P^{\rm NS}_{\overline{Q}Q}(N) + P_{q'q}(N)\,, \tag{2.16}$$

where the non-singlet components are given by

$$\begin{aligned}
&P^{\rm NS}_{QQ}(N) = P^{C_F}_{QQ}(N) + P^{C_A}_{QQ}(N) + P^{n_f}_{QQ}(N)\,, && P^{\rm NS}_{\overline{Q}Q}(N) = P^{C_F}_{\overline{Q}Q}(N) + P^{C_A}_{\overline{Q}Q}(N)\,,\\
&P^{C_F}_{QQ}(N) = C_F^2\left[P_F(N) + \Delta(N)\right]\,, && P^{C_A}_{QQ}(N) = \tfrac{1}{2}C_F C_A P_G(N)\,,\\
&P^{n_f}_{QQ}(N) = n_f C_F T_F P_{NF}(N)\,, && P^{C_F}_{\overline{Q}Q}(N) = C_F^2 P_A(N)\,,\\
&P^{C_A}_{\overline{Q}Q}(N) = -\tfrac{1}{2}C_F C_A P_A(N)\,,
\end{aligned} \tag{2.17}$$

and

$$P_{q'q}(N) = -C_F T_F \frac{8 + 44N + 46N^2 + 21N^3 + 14N^4 + 15N^5 + 10N^6 + 2N^7}{N^3(N+1)^3(N+2)^2(N-1)}\,. \tag{2.18}$$

$P_F(N)$, $\Delta(N)$, $P_G(N)$ and $P_{NF}(N)$ were taken from the appendix of ref. [1] and $P_A(N)$ is given in eq. (5.39) of ref. [4]. We have obtained our explicit expression for $P_{q'q}(N)$ using the equation

$$P_{q'q} = \frac{P^{\rm S}_{QQ} - P^{\rm NS}_{QQ} - P^{\rm NS}_{\overline{Q}Q}}{2n_f}\,, \tag{2.19}$$

where $P^{\rm S}_{QQ}$ is the singlet component[2], calculated in ref. [5]. Equation (2.19) is easily seen to follow from eqs. (2.16) and from eqs. (2.42) of ref. [9].

The expressions for $\hat{a}^{(1)}_Q$ and $d^{(1)}_Q$ are respectively given in eq. (A.12) and (A.13) of ref. [1]. The coefficient $\hat{a}^{(1)}_g$ can be obtained by performing the Mellin transform of the expression $c_{\rm T,g} + c_{\rm L,g}$, where $c_{\rm T,g}$ and $c_{\rm L,g}$ are given in eq. (2.16) of ref. [9]. Thus

$$\begin{aligned}
\hat{a}^{(1)}_g(N,E,\mu) &= C_F\Bigg\{\frac{2(2+N+N^2)}{N(N^2-1)}\log\frac{E^2}{\mu^2} + 4\left[-\frac{2}{(N-1)^2} + \frac{2}{N^2} - \frac{1}{(N+1)^2}\right]\\
&\qquad -2\left[\frac{2}{N-1}S_1(N-1) - \frac{2}{N}S_1(N) + \frac{1}{N+1}S_1(N+1)\right]\Bigg\}\\
d^{(1)}_g(N,\mu_0,m) &= P^{(0)}_{gQ}(N)\log\frac{\mu_0^2}{m^2}\,.
\end{aligned} \tag{2.20}$$

[2] We warn the reader that, sometimes, in the literature, the notation $P^{\rm S}$ is used for the "sea" component, and P_{QQ} is used for the singlet one. Here we use P_{QQ} for the full QQ splitting function.

In order to make a more detailed comparison with our fixed order calculation, we separate the $\mathcal{O}(\bar{\alpha}_s^2)$ contributions to $\sigma(N,E,m)$ according to their colour factors. Choosing for simplicity $\mu=E$ and $\mu_0=m$, and using the notation

$$\hat{a}^{(1)}_{Q/g}(N)=\hat{a}^{(1)}_{Q/g}(N,E,\mu)|_{\mu=E}\,,\quad d^{(1)}_{Q/g}(N)=d^{(1)}_{Q/g}(N,\mu_0,m)|_{\mu_0=m}\,,\tag{2.21}$$

we write

$$\begin{aligned}
\sigma(N,E,m) &= 1+\bar{\alpha}_s(E)\,A(N,E,m)+\bar{\alpha}_s^2(E)\,B(N,E,m)\\
A(N,E,m) &= \hat{a}^{(1)}_Q(N)+d^{(1)}_Q(N)+P^{(0)}_{QQ}(N)\log\frac{E^2}{m^2}\\
B(N,E,m) &= B_{C_F}(N,E,m)+B_{C_A}(N,E,m)+B_{n_f}(N,E,m)+B_{T_F}(N,E,m)\\
B_{C_F}(N,E,m) &= \left\{P^{(0)}_{QQ}(N)\left[d^{(1)}_Q(N)+\hat{a}^{(1)}_Q(N)\right]+P^{C_F}_{QQ}(N)+P^{C_F}_{\overline{Q}Q}(N)\right\}\log\frac{E^2}{m^2}\\
&\quad+\frac{1}{2}\left[P^{(0)}_{QQ}(N)\right]^2\log^2\frac{E^2}{m^2}\\
B_{C_A}(N,E,m) &= \left[P^{C_A}_{QQ}(N)+P^{C_A}_{\overline{Q}Q}(N)+\frac{11}{6}C_A\,d^{(1)}_Q(N)\right]\log\frac{E^2}{m^2}\\
&\quad+\frac{11}{12}C_A\,P^{(0)}_{QQ}(N)\log^2\frac{E^2}{m^2}\\
B_{n_f}(N,E,m) &= \left[P^{n_f}_{QQ}(N)-\frac{2}{3}n_fT_Fd^{(1)}_Q(N)\right]\log\frac{E^2}{m^2}\\
&\quad-\frac{1}{3}n_fT_FP^{(0)}_{QQ}(N)\log^2\frac{E^2}{m^2}\\
B_{T_F}(N,E,m) &= \left[\hat{a}^{(1)}_g(N)\,P^{(0)}_{gQ}+2\,P^{(0)}_{Qg}(N)\,d^{(1)}_g(N)+2\,P_{q'q}(N)\right]\log\frac{E^2}{m^2}\\
&\quad+P^{(0)}_{Qg}(N)\,P^{(0)}_{gQ}(N)\log^2\frac{E^2}{m^2}\,,
\end{aligned}\tag{2.22}$$

where the subscripts C_F, C_A, n_f and T_F denote the C_F^2, C_FC_A, $n_fC_FT_F$ and C_FT_F colour components.

The fixed order calculation of ref. [10] can be used to compute the cross section for the production of a heavy quark pair plus one or two more partons, at order $\bar{\alpha}_s^2$. We separate contributions in which four heavy quarks are present in the final state, from those where a single $Q\overline{Q}$ pair is present together with one or two light partons. These last contributions were computed only in a three-jet configuration, and they are singular in the two-jet limit, that is to say, when $x\to 1$. Furthermore, the virtual corrections to the two-body process $V \to Q + \overline{Q}$ are not included in our calculation. In order to remedy for these problems, we proceed as follows. The $\mathcal{O}(\bar{\alpha}_s^2)$ inclusive cross section for $V \to Q + \overline{Q} + X$, can be written symbolically in the following form

$$\frac{d\sigma}{dx} = a^{(0)}\delta(1-x) + \bar{\alpha}_s \int dY\, a^{(1)}(x,Y) + \bar{\alpha}_s^2 \left[\int dY\, a_l^{(2)}(x,Y) \,+ 2\int dY\, a_h^{(2)}(x,Y)\right] \tag{2.23}$$

where Y denotes all the other kinematical variables, besides x, upon which the final state may depend. We assume $\mu = E$, and we do not indicate, for ease of notation, the dependence upon E and m of the various quantities. The term $a_l^{(2)}$ arises from final states with a single $Q\overline{Q}$ pair plus at most two light partons, while $a_h^{(2)}$ arises from final states with two $Q\overline{Q}$ pairs. The factor of 2 in front of the $a_h^{(2)}$ contribution accounts for the fact that we may detect either one of the two heavy quarks. The moments of the inclusive cross section can be written in the following way

$$\begin{aligned}\sigma(N) = \int dx\, x^{N-1}\frac{d\sigma}{dx} \;&=\; \sigma \,+\, \bar{\alpha}_s \int dx\, dY\, (x^{N-1}-1)\, a^{(1)}(x,Y) \\ &+ \bar{\alpha}_s^2 \left[\int dx\, dY\, (x^{N-1}-1)\, a_l^{(2)}(x,Y) \,+ 2\int dx\, dY\, (x^{N-1}-\tfrac{1}{2})\, a_h^{(2)}(x,Y)\right]\end{aligned} \tag{2.24}$$

where

$$\sigma = a^{(0)} + \bar{\alpha}_s \int dx\, dY\, a^{(1)}(x,Y) + \bar{\alpha}_s^2 \left[\int dx\, dY\, a_l^{(2)}(x,Y) \,+ \int dx\, dY\, a_h^{(2)}(x,Y)\right]\,. \tag{2.25}$$

The expression for $\sigma(N)$ can now be easily computed with our program, since the $(x^{N-1}-1)$ factors regularize the singularities in the two-jet limit, and suppress the two-body $V \to Q + \overline{Q}$ virtual terms. Furthermore, in the massless limit

$$\sigma = 1 + 2\,\bar{\alpha}_s(E) + c\,\bar{\alpha}_s^2(E) \,+ \mathcal{O}\left(\frac{m^2}{E^2}\right) + \mathcal{O}(\bar{\alpha}_s^3) \tag{2.26}$$

where c is a constant. In fact, the $\mathcal{O}(\bar{\alpha}_s^2)$ term does not contain any large logarithm, as long as $\bar{\alpha}_s$ is the coupling with n_f flavours, including the heavy one[3]. Reintroducing the energy and mass dependence, we have

$$\begin{aligned}\sigma(N,E,m) \;&=\; 1 + \bar{\alpha}_s(E)\, L(N,E,m) + \bar{\alpha}_s^2(E)\, M(N,E,m) \\ &\qquad + c\,\bar{\alpha}_s^2(E) + \mathcal{O}\left(\frac{m^2}{E^2}\right) + \mathcal{O}(\bar{\alpha}_s^3)\end{aligned}$$

[3] If instead the cross section formulae are expressed in terms of $\bar{\alpha}_s^{(n_f-1)}$ we have $\sigma = 1 + 2\,\bar{\alpha}_s^{(n_f-1)}(E) + \frac{4}{3}T_F \log\frac{E^2}{m^2}\bar{\alpha}_s^2 + \mathcal{O}(\bar{\alpha}_s^2)$.

$$
\begin{aligned}
L(N,E,m) &= 2+\int dx\,dY\left(x^{N-1}-1\right)a^{(1)}(x,Y) \\
M(N,E,m) &= \int dx\,dY\,\left(x^{N-1}-1\right)a_l^{(2)}(x,Y) \\
&\quad +2\int dx\,dY\,\left(x^{N-1}-\tfrac{1}{2}\right)a_h^{(2)}(x,Y)\;. \qquad (2.27)
\end{aligned}
$$

We have calculated $L(N,E,m)$ and $M(N,E,m)$ numerically, using $E=100$ GeV and $m=8$, 4, 3, 2.5, 2, 1.5, 1, 0.6, 0.5, 0.4, 0.2 GeV, for a vector current coupled to the heavy quark. We expect that, for small masses, $A(N,E,m)$ should coincide with $L(N,E,m)$, and $M(N,E,m)$ should differ from $B(N,E,m)$ by a mass and energy independent quantity, since such term is actually beyond the next-to-leading logarithmic approximation. We find very good agreement between $A(N,E,m)$ and $L(N,E,m)$. We present the results for $M(N,E,m)$ separated into the different colour components

$$
M(N,E,m)=M_{C_F}(N,E,m)+M_{C_A}(N,E,m)+M_{n_f}(N,E,m)+M_{T_F}(N,E,m)\;. \qquad (2.28)
$$

In Fig. 1 we have plotted our results for M_{C_A} (crosses with error bars) and for B_{C_A} (solid lines). An arbitrary (N-dependent) constant has been added to the curves for

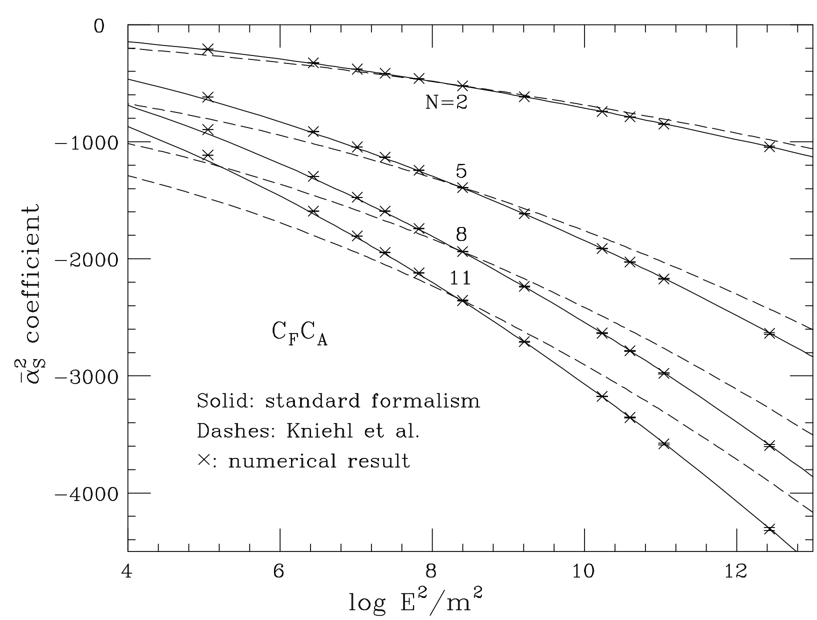

Figure 1: *$C_F C_A$ component of the $\bar{\alpha}_s^2$ coefficient in $\sigma(N,E,m)$, as a function of $\log E^2/m^2$, for $N=$2, 5, 8 and 11.*

B_{C_A}, in order to make them coincide with the numerical result for $m/E=0.015$. We find, as the mass gets smaller, satisfactory agreement for all moments. Notice that, as intuitive reasoning would suggest, for higher moments we need smaller masses to approach the massless limit. In Figs. 2, 3 and 4 we report the analogous results for the remaining colour combinations. Again, we find satisfactory agreement.

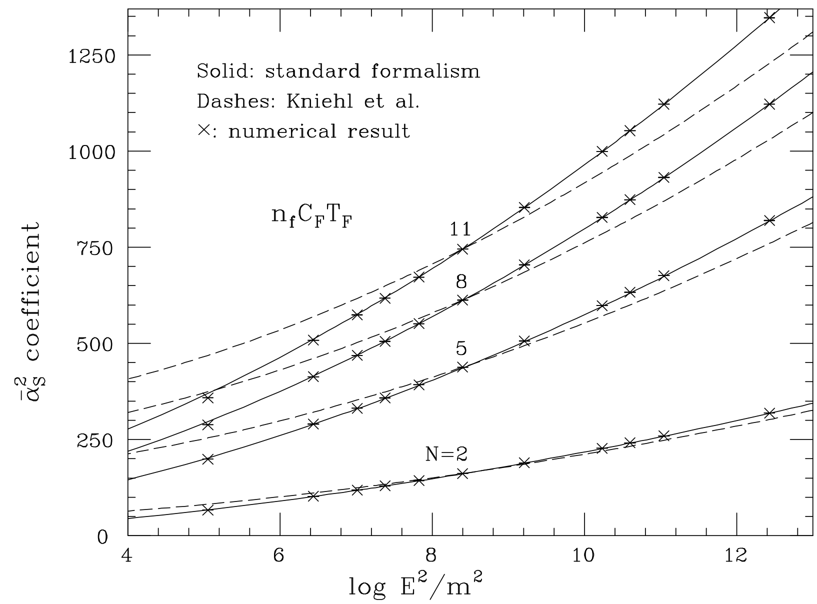

Figure 2: *Same as in Fig. 1, for the $n_f C_F T_F$ component.*

If one follows the procedure proposed by Kniehl et al., eqs. (2.22) are modified in the $C_F C_A$ and in the $n_f C_F T_F$ coefficients. More specifically, the terms proportional to $d_Q^{(1)}$ all disappear from the expressions of the $C_F C_A$ and of the $n_F C_F T_F$ coefficients. In fact, by inspecting formulae (1.11) and (1.12), and the derivation of eq. (2.13), we see that the only relevant difference between the two approaches is that the term $\bar{\alpha}_s(\mu_0)\, d_Q^{(1)}$ is replaced by $\bar{\alpha}_s(\mu)\, d_Q^{(1)}$, which, using the renormalization group equation, amounts to a difference of $-2\pi\, b_0\, \bar{\alpha}_s^2\, d_Q^{(1)} \log \mu^2/\mu_0^2$, precisely what is needed to cancel the term of the same form appearing in eq. (2.13). The modified result of Kniehl et al. is also shown in Figs. 1 and 2 (dashed lines). It is quite clear that their approach does not work.

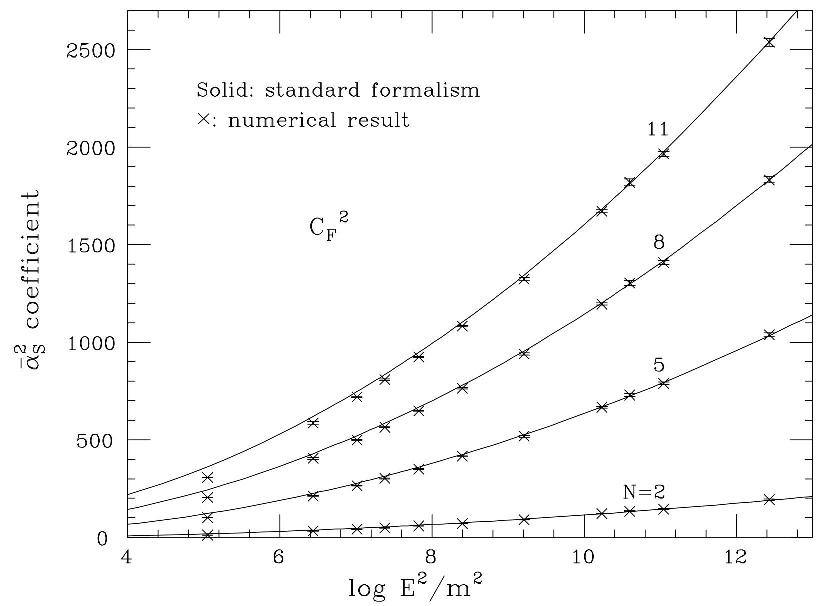

Figure 3: *Same as in Fig. 1, for the C_F^2 component.*

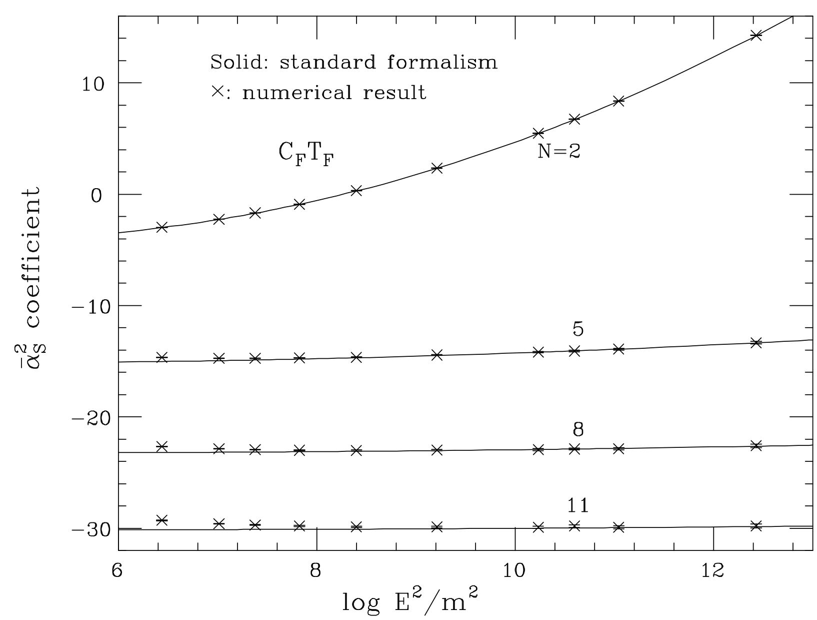

Figure 4: *Same as in Fig. 1, for the $C_F T_F$ component.*

3 Conclusions

In the present work, we have verified at order $\bar{\alpha}_s^2$ the NLO fragmentation function approach to the computation of the heavy quark fragmentation function given in ref. [1]. Besides excluding an alternative that has been proposed in the literature [8], we have given a verification of several ingredients that go into the fragmentation function formalism, such as the initial conditions, computed in ref. [1], the NLO splitting functions in the time-like region [4], and finally we have also performed a further test of the validity of the fixed order calculation of ref. [10].

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